Math 637 (Spring 2023) - Homework 1

1. A theorem of analysis states that every closed subset of $\mathbb{R}^n$ is the zero set of some smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$. Use this theorem to show that if C is any closed subset of $\mathbb{R}^n$, then there is a submanifold of $\mathbb{R}^{n+1}$ such that $X \cap \mathbb{R}^n = C$. Here $\mathbb{R}^n$ is the usual subspace of $\mathbb{R}^{n+1}$ with the last coordinate zero.

2. Show that the height function $(x_1, \cdots, x_n) \to x_n$ on the sphere $\mathbb{S}^{n-1}$ is a Morse function with two critical points.

3. If A and B are disjoint, smooth, closed subsets of a manifold X, prove that there is a smooth function ϕ on X such that $0 \leq \phi \leq 1$ with $\phi = 0$ on A and $\phi = 1$ on B. (Hint: use partition of unity).

4. Use the Brouwer's fixed point theorem to prove the following theorem of Frobenius: if the entries of an $n \times n$ real matrix A are all nonnegative, then A has a real nonnegative eigenvalue.